

Three-dimensional inverse scattering for the wave equation with variable speed: near-field formulae using point sources

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Received 21 January 1988, in final form 25 April 1988

Abstract. We consider the inverse scattering problem for the wave equation with variable speed where the region of interest is probed with waves emanating from point sources. We obtain a three-dimensional trace-type formula, which gives the unknown speed in terms of data and the interior wavefield.

1. Introduction

In this paper we consider the inverse scattering problem for the wave equation with

Here x and y are points in \mathbf{R}^3 , k is a real scalar, and δ is the three-dimensional delta function. The index of refraction $n(x)$ we assume to be a positive, bounded, real-valued function which is identically one outside some bounded region Ω .

We are interested in particular solutions of (2.1) which we specify with the help of the functions

$$G_{\bar{0}}^{\pm}(k, x, y) = G_{\bar{0}}^{\pm}(k, x - y),$$

which satisfy

$$(\nabla^2 + k^2)G_{\bar{0}}^{\pm}(k, x - y) = \delta(x - y). \quad (2.2 \pm)$$

We now specify solutions G^+ and G^- of (2.1) as solutions of the integral equations

$$G^{\pm}(k, x, y) = G_{\bar{0}}^{\pm}(k, x - y) + \int_{\Omega} G_{\bar{0}}^{\pm}(k, x - z)k^2V(z)G^{\pm}(k, z, y) dz \quad (2.3 \pm)$$

where $V = 1 - n^2$.

There are two techniques for showing that (2.3+) and (2.3-) each have unique solutions. One technique [5] shows that for almost every k , (2.3) has a unique solution with $G|V|^{1/2}$ in L^2 . Another technique [6], which uses the limiting absorption principle, shows that for every k , (2.3) has a unique solution in a certain weighted Sobolev space. Both these techniques apply in the present case when V is bounded and has compact support.

Two relations following from (2.3) will be needed in §4: first, that G^- is the

Proof. The proof, based on the use of Green's formula, is similar to the corresponding proof in [3] and is omitted here.

$$\int_{\partial\Omega} \left(G_0^-(z-x) \frac{\partial}{\partial\nu} G_0^+(z-y) - G_0^+(z-y) \frac{\partial}{\partial\nu} G_0^-(z-x) \right) dS_z = G_0^-(y-x) - G_0^+(x-y). \tag{2.6}$$

Proof. We set $n(x)$ equal to one in theorem 1.

Corollary 2. Suppose the hypotheses of theorem 1 hold and $G^\pm = G_0^\pm + G_{sc}^\pm$. Then

$$G_{sc}^-(y, x) - G_{sc}^+(x, y) = \int_{\partial\Omega} \left(G_0^-(z-x) \frac{\partial}{\partial\nu} G_{sc}^+(z, y) - G_{sc}^+(z, y) \frac{\partial}{\partial\nu} G_0^-(z-x) \right) dS_z$$

Proof. We substitute $C^+ = C^+ + C^+$ into (2.1), simplify, and evaluate at $k=0$

Proof of Lemma 2. Following [1] we apply our present information to the construction of

- [1] Rose J H and Cheney M 1987 Self-consistent equations for variable velocity three-dimensional inverse scattering *Phys. Rev. Lett.* **59** 954-7
- [2] Cheney M, Rose J H and DeFazio B 1988 A fundamental integral equation of scattering theory *SIAM*